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NEW APPROXIMATE ANALYTIC SOLUTIONS OF THE EQUATIONS OF GAS DYNAMICS†

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Two-term approximate solutions of the equations of gas dynamics in variables of the velocity hodograph are constructed both for subsonic flow containing an arbitrary analytic function of the complex variable and a sonic (critical) point and for supersonic motion having two arbitrary functions of the characteristic variables and the sonic point. The solutions are constructed without using the method of approximations. These solutions are matched on the sonic line for transonic motion using an example.

1. FUNDAMENTAL FORMULAE

IT IS WELL known [1], that the velocity potential φ and the stream function ψ for the steady-state adiabatic motion of an ideal gas in the hodograph variables τ, θ satisfy a non-symmetrical linear system of partial differential equations with variable coefficients

$$\frac{\partial\varphi}{\partial\tau}+P\frac{\partial\psi}{\partial\theta}=0,\quad \frac{\partial\varphi}{\partial\theta}-Q\frac{\partial\psi}{\partial\tau}=0,$$
$$P=\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}},\quad Q=\frac{2\tau}{(1-\tau)^{\beta}} \tag{1.1}$$
$$\beta=\frac{1}{\gamma-1},\quad \tau=\frac{\upsilon^2}{\upsilon_{max}^2}$$

where υ is the modulus of the velocity, υ_{max} is the limiting velocity of the flow, and γ is the Poisson adiabatic index (for atmospheric air $\gamma=1.4025$).

We change to physical coordinates x, y by means of the formula

$$dz=(d\varphi+i\rho_0\rho^{-1}d\psi)\upsilon^{-1}e^{i\theta}\qquad (z=x+iy) \tag{1.2}$$
$$\rho=\rho_0(1-\tau)^{\beta},\quad M=\frac{\upsilon}{a_*}=[2\beta\tau/(1-\tau)]^{1/2}$$

(ρ is the density, ρ_0 is the characteristic density, and a_* is the velocity of sound).

In system (1.1) a change is usually made [2, 3] to the new variable $\sigma=\sigma(\tau)$ and one reduces it to a symmetrical form containing a single coefficient, namely, the Chaplygin function $K=PQ$, which depends implicitly on σ.

We will construct approximate solutions for the system of equations (1.1).

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2. SUBSONIC FLOW

For subsonic motion ($P>0,\ 0\leqslant\tau<\tau_*$) Eqs (1.1) are of the elliptic type. The critical point at which the velocity of the flow is equal to the velocity of sound corresponds to the value $\tau_*=(\gamma-1)/(\gamma+1)$.

An exact general solution and corresponding multiterm jet boundary-value problems for φ and ψ were constructed [1] by the method of separation of variables for system (1.1) in the form of converging series containing Gauss hypergeometric functions. Accurate general solutions for a symmetrical system of equations of gas dynamics in σ, θ variables were proposed [5] in the form of integral and differential series containing an arbitrary analytic function of the complex variable.

Different approximate solutions for the symmetrical equations of gas dynamics are based on an approximation of the Chaplygin function [3, 6].

We will construct an approximate solution for the system of equations (1.1) in the form of a two-term integral operator [5]

$$\varphi=c_1\theta-c_2\int_{\tau_*}^{\tau}\mathbf{P}d\tau+\operatorname{Re}[\alpha(\tau)w(\zeta)+A\int w(\zeta)d\zeta]$$

$$\psi=c_1\int_{\tau_*}^{\tau}\frac{d\tau}{\mathbf{Q}}+c_2\theta+\operatorname{Im}[\beta(\tau)w(\zeta)+B\int w(\zeta)d\zeta] \tag{2.1}$$

$$\zeta=s+i\theta,\quad \left(s=\int_{\tau}^{\tau_*}\left(\frac{\mathbf{P}}{\mathbf{Q}}\right)^{1/2}d\tau\right)$$

Here $\alpha(\tau)$, $\beta(\tau)$ are continuous functions of a single variable τ, A and B are constants, and $w(\zeta)$ is an arbitrary analytic function of the complex argument ζ.

Introducing the corresponding derivatives of (2.1) and (1.1) and using the well-known properties of analytical functions of a complex variable, we can write

$$\operatorname{Re}\Delta_1(\tau)w-\left(\frac{\mathbf{P}}{\mathbf{Q}}\right)^{1/2}\Delta(\tau)w'=0,\quad \operatorname{Im}\Delta_2(\tau)w+\Delta(\tau)w'=0 \tag{2.2}$$

$$\Delta_1(\tau)=\alpha'-A\left(\frac{\mathbf{P}}{\mathbf{Q}}\right)^{1/2}-B\mathbf{P}$$

$$\Delta_2(\tau)=A+\mathbf{Q}\beta'-B(\mathbf{PQ})^{1/2}$$

$$\Delta(\tau)=\alpha-\beta(\mathbf{PQ})^{1/2}$$

These equations are satisfied identically for an arbitrary function $w(\zeta)$ if we impose the following conditions

$$\Delta_1(\tau)=0,\quad \Delta_2(\tau)=0,\quad \Delta(\tau)=0 \tag{2.3}$$

The functions α and β can be determined, apart from additive constants of integration, from the first two differential equations (2.3). These constants can be omitted as unimportant for the operator (2.1) if, conversely, we use them so that at the critical point τ_* the equations $\alpha(\tau_*)=\beta(\tau_*)=0$ are satisfied. Then, for any index γ we obtain the expressions

$$\alpha=-BI_1+AI_3,\quad \beta=-AI_2+BI_3 \tag{2.4}$$

Henceforth we will assume $\gamma=1.4$ in the calculations. The integrals that occur here can be expressed in the following analytic form

$$I_1=\int_{\tau_*}^{\tau} P_{d\tau}=t^{-1}+\frac{t^{-3}}{3}-t^{-5}-\mathrm{Arth}\,t\Big|_{t_*}^{t}$$

$$I_2=\int_{\tau_*}^{\tau}\frac{d\tau}{Q}=t+\frac{t^3}{3}+\frac{t^5}{5}-\mathrm{Arth}\,t\Big|_{t_*}^{t}$$

$$I_3=\int_{\tau_*}^{\tau}\left(\frac{P}{Q}\right)^{1/2}d\tau=\sqrt{6}\,\mathrm{Arch}\sqrt{\frac{6}{5}}t-\mathrm{Arth}\frac{\sqrt{6t^2-5}}{t}\Bigg|_{t_*}^{t}$$

$$t=\sqrt{1-\tau} \qquad (2.5)$$

All the functions (2.5) are strictly negative and increase monotonically from $-\infty$ to 0, while the function $s=-I_3$ decreases from ∞ to 0.

The constants A and B in (2.4) can be chosen so that the parameter $\Delta(\tau)$ (2.3) over the whole interval $0\leqslant\tau\leqslant\tau_*$ is close to zero: $\Delta(\tau)\approx 0$, and any deviation from zero can be neglected in practice. The approximate solution (2.1), (2.4) of system (1.1) obtained in this way is not encountered in the literature.

The values of the functions α, β, s, Δ and M are calculated to the sixth decimal place for $A=0.829223$ and $B=0.866714$ (the values given in Table 1 are accurate to the fourth place). The functions $\alpha(\tau)<0$ and $\beta(\tau)>0$. Here $\alpha(\tau_*)=\beta(\tau_*)=0$ and $\alpha(0)=\beta(0)=0$ (infinite values of the quantities (2.5) at the stagnation point $\tau=0$ are reduced to zero in (2.4)).

When solving direct boundary-value problems we will take the function $w(\zeta)$ in the form of a converging exponential series

$$w(\zeta)=\Sigma(A_ne^{n\omega\zeta}+B_ne^{-n\omega\zeta}) \qquad (2.6)$$

in which ω, A_n and B_n are real constants, found from the boundary conditions. Here and henceforth summation is carried out from $n=1$ to $n=\infty$.

Taking (2.6) into account we can write (2.1) in the form

$$\varphi=c_1\theta-c_2\int_{\tau_*}^{\tau}P\,d\tau+\Sigma\{A_n[\alpha+(n\omega)^{-1}A]e^{n\omega s}+B[\alpha-(n\omega)^{-1}A]e^{-n\omega s}\}\cos n\omega\theta$$

$$\psi=c_1\int_{\tau_*}^{\tau}Q^{-1}d\tau+c_2\theta+\Sigma\{A_n[\beta+(n\omega)^{-1}B]e^{n\omega s}+B_n[B(n\omega)^{-1}-\beta]e^{-n\omega s}\}\sin n\omega\theta \qquad (2.7)$$

TABLE 1

τ	$M\times10^3$	$-\alpha\times10^4$	$\beta\times10^4$	$-s$	$\Delta\times10^4$
0	0	0	0	∞	0
0.01	225	47	250	1.141741	203
0.03	393	230	433	0.618688	200
0.05	513	287	491	0.390833	196
0.07	613	295	500	0.251754	178
0.09	703	270	473	0.157140	155
0.11	786	219	413	0.090168	123
0.13	864	146	318	0.043025	81
0.15	939	59	178	0.012218	32
1/6	1000	0	0	0	0

We will consider, as the simplest example in the extended formulation, the classical problem [1] of a symmetrical flow of gas from a slit of width $2H$ in a vessel bounded by two plane semi-infinite walls, which make an angle of 2λ with each other. Free jets break away from the edges of the slit with a velocity equal to the velocity of sound, moving in the down flow direction to infinity where the width of their cross-section is equal to $2h$. The origin of coordinates is on the axis of symmetry ox in the cross-section of the slit.

In view of the symmetry we will consider the lower part of the flow in which $\theta > 0$. This flow is bounded by the streamline $\psi = -Q$ (Q is half the gas flow rate) and $\psi = 0$ is the line of symmetry of the flow.

We have the following boundary conditions

$$\begin{aligned} &\psi = 0: && \theta = 0, && 0 \leqslant \tau \leqslant \tau_* \\ &\psi = -Q: && \theta = \lambda, && 0 \leqslant \tau \leqslant \tau_* \\ &\psi = -Q: && 0 \leqslant \theta \leqslant \lambda, && \tau = \tau_* \end{aligned} \tag{2.8}$$

To solve the problem we will write (2.7) with $c_1 = 0$ and $A_n = 0$ ($n = 1, 2, \ldots$)

$$\begin{aligned} \varphi &= -c_2 \int_{\tau_*}^{\tau} P d\tau + \Sigma B_n [\alpha - (n\omega)^{-1} A] e^{-n\omega s} \cos n\omega\theta \\ \psi &= c_2\theta + \Sigma B_n [B(n\omega)^{-1} - \beta] e^{-n\omega s} \sin n\omega\theta \end{aligned} \tag{2.9}$$

Assuming $c_2 = -Q/\lambda$, $\omega = \pi/\lambda$ we can satisfy the first two conditions of (2.8). We can also satisfy the third condition if we expand the function $f(\theta) = Q(\theta/\lambda - 1)$ in a Fourier series in $\sin n\omega\theta$ in the interval $0 \leqslant \theta \leqslant \lambda$ and then obtain B_n in the usual way

$$B_n = 2Q(B\lambda)^{-1}\left[(-1)^n\left(1 - \frac{\lambda}{n\pi}\right) - 1\right] \quad n = 1, 2, \ldots \tag{2.10}$$

Note that the function ψ in (2.9) when $c_2 = 0$ and

$$B_n = 2Q\left(1 - \cos\frac{n\pi\mu}{\lambda}\right)(B\lambda)^{-1} \tag{2.11}$$

will represent the problem of a symmetrical stream flow of a jet of gas around a wedge having velocities equal to the velocity of sound at infinity and in jets separated from the ends of the wedge.

In (2.11) λ is the angle of the upper side of the wedge and μ is the angle of the free jet in the downflow at infinity, which they make with the axis of symmetry ox.

Note that at the critical point $\tau = \tau_*$ ($\alpha(\tau_*) = \beta(\tau_*) = s(\tau_*) = 0$) for problem (2.8) and (2.9), φ and ψ are given by

$$\begin{aligned} &\varphi(\tau_*, \theta) - A\Sigma(n\omega)^{-1} B_n \cos n\omega\theta \\ &\psi(\tau_*, \theta) = c_2\theta + B\Sigma(n\omega)^{-1} B_n \sin n\omega\theta \end{aligned} \tag{2.12}$$

3. PARADOX OF THE APPROXIMATION

The equality $\Delta(\tau) = 0$ (2.3) can be satisfied exactly if we replace the quantities P and Q in it, for example, by the comparison functions P_1 and Q_1, given by

$$P \approx P_1 = a\tau^{-p}, \quad Q \approx Q_1 = b\tau^{q} \quad \left(q = \frac{4-p}{3}\right) \tag{3.1}$$

(a, b and p are positive numbers).

Introducing (3.1) into (2.3) and dropping the constants of integration as unimportant, we obtain the following equalities for the functions (2.4) and $\Delta(\tau)$

$$I_1 = \frac{a\tau^{1-p}}{1-p}, \quad I_2 = \frac{\tau^{1-q}}{b(1-q)}, \quad (PQ)^{1/2} = (ab)^{1/2}\tau^{\frac{q-p}{2}}$$
$$I_3 = 2\left(\frac{a}{b}\right)^{1/2} \frac{\tau^{1-\frac{p+q}{2}}}{2-p-q} \tag{3.2}$$

$$\Delta(\tau) = A\left(\frac{a}{b}\right)^{1/2} \tau^{1-\frac{p+q}{2}}\left(\frac{2}{2-p-q} + \frac{1}{1-q}\right) - Ba\tau^{1-p}\left(\frac{1}{1-p} + \frac{2}{2-p-q}\right) = 0$$

The equation $\Delta(\tau) = 0$ is satisfied when

$$B = 0, \qquad q = (4-p)/3 \tag{3.3}$$

This case is convenient for approximating the two functions P and Q.

Two other cases, namely, (1) $aB = A(a/b)^{1/2}$, $p = q$ and (2) $A = 0$, $q = 4 - 3p$ are unsuitable for such an approximation.

The continuous curves in Fig. 1 represent graphs of the functions P and Q (1.1), and the dashed curves are graphs of the functions (3.1) for case (3.3) with $p = 1.22$, $a = 0.20$ and $b = 2.08$. The values of Q are plotted multiplied by 10^3. It can easily be shown by calculation that a peculiar approximation paradox arises here consisting of the fact that in the section $0.05 \leqslant \tau \leqslant 0.11$, in which the approximations of the functions P and Q by functions (3.1) are satisfactory, the parameter $\Delta(\tau)$ (2.3) under the same conditions (the constants of integration are dropped) for exact values of P and Q differs considerably from zero, and this parameter increases monotonically from $\Delta(0) = -\infty$ to $\Delta(\tau_*) = 0$. In the section considered it takes values of $\Delta(0.05) = -1.1$ and $\Delta(0.11) = -0.42$, i.e. it differs considerably from zero. This discrepancy can obviously occur if the approximation (3.1) goes outside the framework of the classification of the group properties of differential equations established in [7] for the Chaplygin function, which occurs in the second-order equation for the stream function ψ.

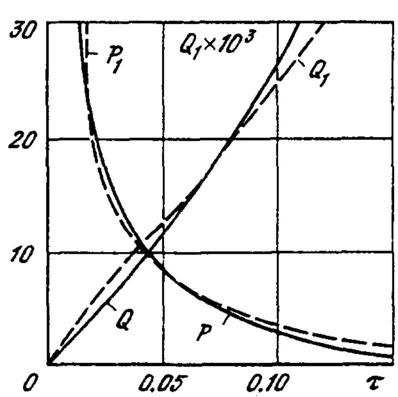

Fig. 1.

This paradox can be eliminated by putting, for example, the constant $A = 0.01$, on which, however, the approximation does not depend. In this case, the need for approximation (3.1) generally disappears, and also the need to consider the case (3.3) in hydrodynamics. On the other hand, an additional "mechanism" for estimating the approximation arises.

4. SUPERSONIC FLOW

In the supersonic flow of a gas ($P < 0$, $\tau_* \leqslant \tau < \tau_1$) the system of equations (1.1) becomes hyperbolic

$$\frac{\partial\varphi}{\partial\tau} - \chi\frac{\partial\psi}{\partial\theta} = 0, \quad \frac{\partial\varphi}{\partial\theta} - Q\frac{\partial\psi}{\partial\tau} = 0$$
$$\chi = -P = \frac{(2\beta+1)\tau - 1}{2\tau(1-\tau)^{\beta+1}}, \quad Q = \frac{2\tau}{(1-\tau)^{\beta}} \tag{4.1}$$

For symmetrical equations of supersonic gas dynamics we know (for example, [2, 3, 6]) approximate solutions obtained using the Chaplygin-function approximation, which depend implicitly on $\sigma = \sigma(\tau)$.

By analogy with (2.1) we will seek a solution of the system of equations (4.1) in the form of a finite integral operator

$$\varphi = c_1\theta + c_2\int_{\tau_*}^{\tau} \chi d\tau + \gamma(\tau)[f(\xi) + F(\eta)] + E[\int f(\xi)d\xi + \int F(\eta)d\eta]$$
$$\varphi = c_1\int_{\tau_*}^{\tau} Q^{-1}d\tau + c_2\theta + \delta(\tau)[f(\xi) - F(\eta)] + D[\int f(\xi)d\xi - \int F(\eta)d\eta] \tag{4.2}$$
$$\xi = \sigma - \theta, \quad \eta = \sigma + \theta \quad \left(\sigma = \int_{\tau_*}^{\tau}\left(\frac{X}{Q}\right)^{1/2} d\tau\right)$$

where c_1, c_2, E and D are constants, and $f(\xi)$, $F(\eta)$ are arbitrary functions of the characteristic variables ξ, η.

The functions $\gamma(\tau)$, $\delta(\tau)$ must be chosen in such a way that formulae (4.2) satisfy system (4.1) for arbitrary f and F. We will introduce into (4.1) the corresponding derivatives of (4.2).

$$(\gamma' + E\sigma' + D\chi)(f + F) + \sigma'[\gamma + \delta(\chi Q)^{1/2}](f' + F') = 0$$
$$(E + Q\delta' + DQ\sigma')(f - F) + [\gamma + \delta(\chi Q)^{1/2}](F' - f') = 0 \tag{4.3}$$

These equations are satisfied identically if γ and δ are subject to the conditions

$$\gamma = -E\sigma - D\int_{\tau_*}^{\tau} Xd\tau, \quad \delta = -E\int_{\tau_*}^{\tau} Q^{-1}d\tau - D\sigma \tag{4.4}$$

$$\Delta(\tau) = \gamma + \delta(\chi Q)^{1/2} = 0 \tag{4.5}$$

Here, for the case when $\psi = 1.4$ we have

$$\int_{\tau_*}^{\tau} \chi d\tau = -\left(t^{-1} + \frac{t^{3}}{3}\right) + t^{-5} + \operatorname{Arth} t \,\Big|_{t_*}^{t}$$

$$\int_{\tau_*}^{\tau} Q^{-1}d\tau = t + \frac{t^3}{3} + \frac{t^5}{5} - \text{Arth}\,t\Big|_{t_*}^{t}$$

$$\sigma = \int_{\tau_*}^{\tau} \left(\frac{\chi}{Q}\right)^{1/2} d\tau = \sqrt{6}\,\text{arctg}\sqrt{\frac{6\tau-1}{6(1-\tau)}} - \text{arctg}\sqrt{\frac{6\tau-1}{1-\tau}}\Bigg|_{\tau_*}^{\tau} \tag{4.6}$$

$$t = \sqrt{1-\tau}, \quad \sigma > 0$$

A calculation of expression (4.5) and (4.6) to six places of decimals showed that for $E = 0.086021$ and $D = 0.034710$ in the range $\tau_* \leqslant \tau \leqslant 0.34$ we have $|\Delta_{max}(\tau)| \leqslant 0.0019$.

In Table 2, for these values of τ, the parameter $\Delta(\tau)$ is negative, and at the points $\tau_1 = 0.28$ and $\tau_2 = 0.30$ are positive, and the function $\alpha < 0$.

Over this section the solution (4.2), (4.4) can also be used in practice as an approximate solution for the system of equations (4.1).

It is easy to show that if χ and Q in the section $0.26 \leqslant \tau \leqslant 0.416$ are replaced by the hypothetical expressions

$$\chi \approx \chi_1 = a_1\tau^p, \quad Q \approx Q_1 = b_1\tau^q, \quad \left(q = \frac{p+4}{3}\right) \tag{4.7}$$

then $\Delta(\tau) = 0$ everywhere, and for $D = 0$, $p = 2.853646$ ($q = 2.284549$), $a_1 = 144.3216$ and $b_1 = 23.9579$ the graphs of the functions χ and Q and the functions (4.7) are close to one another (Fig. 2—the scale for Q is increased by a factor of five).

Here, as in the case of subsonic motion, the approximation does not depend on the constant E and a similar paradox arises, that is, the parameter $\Delta(\tau)$ (in the section of satisfactory approximation) differs considerably from zero, being calculated for exact values of χ and Q.

When solving direct boundary-value problems in the characteristic variables one can use, for example, the method described in [3]. We will not dwell on this here.

We will consider an example of a continuous solution with the possibility of a transition (matching) through a critical point. For this we will take f and F in the form

$$f = \frac{1}{2}\Sigma(a_n \sin n\Omega\xi + b_n \cos n\Omega\xi)$$

$$F = \frac{1}{2}\Sigma(a_n \sin n\Omega\eta + b_n \cos n\Omega\eta)$$

TABLE 2

τ	M	$-\alpha \times 10^4$	$\beta \times 10^4$	$\sigma \times 10^4$	$-\Delta \times 10^4$
1/6	1	0	0	0	0
0,18	1,05	24	18	79	14
0,20	1,12	51	37	296	19
0,22	1,19	75	50	568	15
0,24	1,26	97	59	873	7
0,26	1,33	120	65	1199	5
0,28	1,39	145	67	1541	–4
0,30	1,46	172	68	1893	–4
0,32	1,57	204	66	2253	1
0,34	1,61	240	63	2619	16
0,36	1,68	282	59	2991	41
0,38	1,75	331	53	3367	79
0,40	1,83	388	46	3747	135

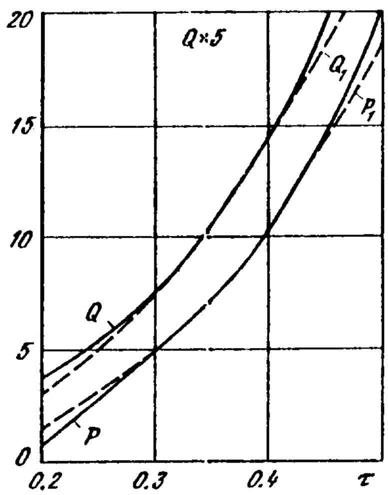

FIG. 2.

(Ω, a_n and b_n are constants, $n=1, 2, \ldots$). Then (4.2) with $c_1=0$ take the form

$$\begin{aligned}\varphi &= c_2 \int_{\tau_*}^{\tau} \chi d\tau + \Sigma\{a_n[\gamma \sin n\Omega\sigma - E(n\Omega)^{-1}\cos n\Omega\sigma]+ \\ &+b_n[\gamma \cos n\Omega\sigma + E(n\Omega)^{-1}\sin n\Omega\sigma]\}\cos n\Omega\theta \\ \psi &= c_2\theta + \Sigma\{b_n[\delta \sin n\Omega\sigma - D(n\Omega)^{-1}\cos n\Omega\sigma]- \\ &-a_n[\delta\cos n\Omega\sigma + D(n\Omega)^{-1}\sin n\Omega\sigma]\}\sin n\Omega\theta\end{aligned} \tag{4.8}$$

If we put $c_2=-Q/\lambda$, $\Omega=\pi/\lambda$ here, the function ψ will satisfy the first two conditions of (2.8). We will take into account the fact that $\gamma(\tau_*)=\delta(\tau_*)=\sigma(\tau_*)=0$, and equating expressions (2.12) and (4.8) to one another (matching) when $\tau=\tau_*$, we obtain the following relation (succession) between the constants

$$a_n = AB_n/E, \quad b_n=-BB_n/D \quad (n=1,2,\ldots)$$

which ensures that the third condition of (2.8) is satisfied and, at the same time, enables us to consider transonic motion in the neighbourhood of the sonic line, and then extend the investigation into the supersonic part of the flow in characteristic variables using the scheme described in [3, 6, 8].

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